UNIVERSITY OF PENNSYLVANIA

A PROBLEM IN TRANSIENT PHENOMENA IN COUPLED CIRCUITS

ENOS EBY WITMER

A THESIS

IN PHYSICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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A PROBLEM IN TRANSIENT PHENOMENA IN COUPLED CIRCUITS.*

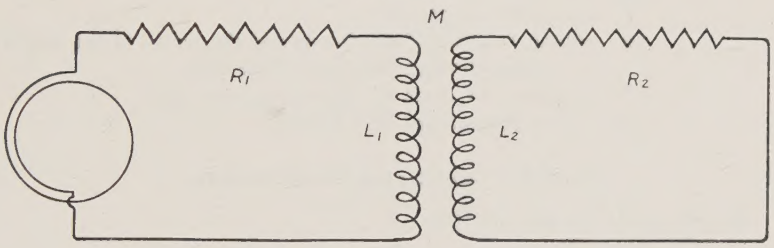
BY

ENOS E. WITMER.

PART I.

THERE is a device in common use for measuring the velocity of magnetized projectiles which records the instant of passage of the projectile through a circular coil by means of a chronographic record of the currents induced in a circuit in series with or magnetically coupled with that coil. Since these circuits contain inductance, the currents will not in general be in phase with the electromotive force, which means that there will be a small time error in the determination of the instant of passage from the

FIG. 1.



current curves. For accurate work it is desirable to know these time differences, which we shall designate time lags.

Since the fundamental differential equations for coupled circuits are well known, this problem can be solved mathematically. It is a simple matter to obtain expressions for the currents in terms of a single type of integral, but the evaluation of the integral offers difficulties. The present paper deals with the evaluation of this integral and with the approximations for special cases which will show the general trend of the phenomena as certain parameters vary.

* Communicated by Dr. C. B. Bazzoni. Contribution from the Randal Morgan Laboratory of Physics.

The problem may be formulated as follows: Two electrical circuits containing only resistance and inductance are magnetically coupled by an air-core transformer. In the primary is a circular coil. An ideal bar magnet, whose axis coincides with the axis of the coil, moves along this axis with a uniform velocity, V, thus generating a momentary electromotive force. The problem is to determine the variation with time of the electromotive force and the currents in the primary and secondary: A diagram of the circuits is given in Fig. 1.

Notation:

E = e.m.f. induced in the circular coil;
i_1 = current in primary;
i_2 = current in secondary;
R_1 = resistance of primary;
R_2 = resistance of secondary;
L_1 = self-inductance of primary;
L_2 = self-inductance of secondary;
M = mutual inductance;
$2a$ = length of bar magnet;
b = radius of circular coil;
V = velocity of magnet;
t = time, measured from the instant the centre of the magnet passes through the plane of the coil;
N = number of turns of wire in circular coil;
m = pole strength of the magnet.

THE E.M.F. CURVE AND THE CURRENTS.

It can easily be shown that

$$E = f(t) \equiv p \left\{ \frac{1}{[(Vt-a)^2+b^2]^{\frac{3}{2}}} - \frac{1}{[(Vt+a)^2+b^2]^{\frac{3}{2}}} \right\}, \tag{1}$$

where, if E is to be in volts,

$$p = \frac{2\pi N b^2 V m}{10^8} \tag{2}$$

when c. g. s. units are used.

Let

$$\left.\begin{aligned} c &= \frac{a}{b} \\ u &= \frac{Vt}{b} \end{aligned}\right\}. \tag{3}$$

Then

$$E = \frac{p}{b^3}\varphi(u) \equiv \frac{p}{b^3}\left\{ \frac{1}{U_1^3} - \frac{1}{U_2^3} \right\}, \tag{4}$$

where

$$\left.\begin{aligned} U_1 &= \sqrt{(u-c)^2+1} \\ U_2 &= \sqrt{(u+c)^2+1} \end{aligned}\right\}. \tag{5}$$

The general shape of the e.m.f. curve is the same as that of the i_1, u curve in Fig. 2 (c), which was plotted for $c = 1.174$, the value based on the apparatus used in the experimental tests.

The differential equations for the currents are the well-known equations for coupled circuits:

$$L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + i_1 R_1 = f(t) = E, \tag{6}$$

$$L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} + i_2 R_2 = 0. \tag{7}$$

Introducing the condition

$$i_1 = i_2 = 0 \text{ when } t = \pm \infty,$$

the solutions of (6) and (7) are

$$i_1 = \frac{\sigma}{\alpha_0 - \beta_0}\left\{(\alpha_0 L_2 - R_2)\, e^{-\alpha_0 t}\int_{-\infty}^{t} e^{\alpha_0 \tau} f(\tau)\, d\tau - (\beta_0 L_2 - R_2)\, e^{-\beta_0 t}\int_{-\infty}^{t} e^{\beta_0 \tau} f(\tau)\, d\tau\right\}, \tag{8}$$

$$i_2 = \frac{\sigma M}{\alpha_0 - \beta_0}\left\{\beta_0 e^{-\beta_0 t}\int_{-\infty}^{t} e^{\beta_0 \tau} f(\tau)\, d\tau - \alpha_0 e^{-\alpha_0 t}\int_{-\infty}^{t} e^{\alpha_0 \tau} f(\tau)\, d\tau\right\}. \tag{9}$$

Where

$$\sigma = \frac{1}{L_1 L_2 - M^2}, \tag{10}$$

$$k = \frac{M}{\sqrt{L_1 L_2}}, \tag{11}$$

$$\alpha_0, \beta_0 = \frac{1}{2(1-k^2)}\left\{\frac{R_1}{L_1} + \frac{R_2}{L_2} \pm \sqrt{\left(\frac{R_1}{L_1} - \frac{R_2}{L_2}\right)^2 + 4k^2 \frac{R_1}{L_1}\cdot\frac{R_2}{L_2}}\right\}. \tag{12}$$

Equation (12) shows that α_0 and β_0 are always positive. Take $\alpha_0 \geq \beta_0$.

Let

$$\left.\begin{aligned} \frac{R_1}{L_1}\frac{b}{V} &= \alpha_1 \\ \frac{R_2}{L_2}\frac{b}{V} &= \alpha_2 \\ \frac{\alpha_0 b}{V} &= \alpha \\ \frac{\beta_0 b}{V} &= \beta \end{aligned}\right\}. \tag{13}$$

By means of transformations (3) and (13), equations (8), (9) and (12) can be put into the following form:

$$\alpha, \beta = \frac{1}{2(1-k^2)} \left\{ \alpha_1 + \alpha_2 \pm \sqrt{(\alpha_1 - \alpha_2)^2 + 4k^2\alpha_1\alpha_2} \right\}, \tag{14}$$

$$i_1 = \frac{k_1}{\alpha - \beta} \Big\{ (\alpha - \alpha_2)[I(\alpha, u-c) - I(\alpha, u+c)] - (\beta - \alpha_2)[I(\beta, u-c) - I(\beta, u+c)] \Big\}. \tag{15}$$

$$i_2 = \frac{k_2}{\alpha - \beta} \Big\{ \beta[I(\beta, u-c) - I(\beta, u+c)] - \alpha[I(\alpha, u-c) - I(\alpha, u+c)] \Big\}. \tag{16}$$

Where

$$I(\lambda, u) = e^{-\lambda u} \int_{-\infty}^{u} e^{\lambda y} \frac{dy}{(y^2+1)^{\frac{3}{2}}}, \tag{17}$$

$$k_1 = \frac{q}{L_1(1-k^2)}, \tag{18}$$

$$k_2 = \frac{qk}{\sqrt{L_1 L_2}(1-k^2)}, \tag{19}$$

and

$$q = \frac{2\pi Nm}{10^8}. \tag{20}$$

We see from (15) and (16) the shape of the i, u curves is a function of the parameters α_1, α_2, k, and c, since α and β are functions of α_1, α_2, and k.

The problem is thus reduced to the evaluation of a one-parameter integral.

METHODS OF INTEGRATION.

The chief difficulty in this problem is evaluating the integral (17). We give five methods adapted to different values of λ.

The First Method: For large values of λ the best integral formula for evaluating (17) is the following one obtained by successive partial integration:

$$e^{-\lambda u} \int_{-\infty}^{u} e^{\lambda y} \frac{dy}{(y^2+1)^{\frac{3}{2}}} = \frac{1}{\lambda} \cdot \frac{1}{(u^2+1)^{\frac{3}{2}}} + \frac{1}{\lambda^2} \cdot \frac{3u}{(u^2+1)^{\frac{5}{2}}} + \frac{1}{\lambda^3} \cdot \frac{3(4u^2-1)}{(u^2+1)^{\frac{7}{2}}} + \frac{1}{\lambda^4} \cdot \frac{15u(4u^2-3)}{(u^2+1)^{\frac{9}{2}}} + \frac{1}{\lambda^4} e^{-\lambda u} \int_{-\infty}^{u} e^{\lambda y} \frac{45(1-12y^2+8y^4)\,dy}{(y^2+1)^{\frac{11}{2}}} \tag{21}$$

The ratio, R, of the integrand of the integral on the right, to that of the integral on the left, is

$$R = \frac{45\,(1 - 12y^2 + 8y^4)}{\lambda^4\,(y^2+1)^4}.$$

The maximum numerical value of R is $R = \frac{45}{\lambda^4}$. Hence if λ is large enough, the last integral is negligible. If $\lambda > 8.2$ the integrand of the last integral is everywhere less than 1 per cent. of the integrand of the original integral. It can also be shown that for

$$|y| > 1.704\,,\ R < \frac{6.74}{\lambda^4};$$

so that for

$$\lambda > 5.1\,,\ u < -1.704$$

the integrand of the last integral is everywhere less than 1 per cent. of the integrand of the original integral.

The Second Method: The second, third, and fourth methods, though valid for all values of λ and u, do not lend themselves to numerical computation with equal facility in all cases. The second method is best adapted to values of λ just too small to use the first method.

By a combination of partial integration and expansion in series,

$$\begin{aligned} e^{-\lambda u}\int_{-\infty}^{u} e^{\lambda y}\frac{dy}{(y^2+1)^{\frac{3}{2}}} &= \frac{1}{\lambda}\cdot\frac{1}{(u^2+1)^{\frac{3}{2}}} \\ &+ \frac{3}{\lambda}\,e^{-\lambda u}\Big\{\sum_{M=0}^{\infty} c_M u^M\,(u^2+1)^{-\frac{3}{2}} \\ &+ c_{-1}\log\left(\sqrt{u^2+1}+u\right) + g_1(\lambda)\Big\}, \end{aligned} \tag{22}$$

Where

$$c_M = \frac{\lambda^M}{M!}\cdot\frac{1}{(M-3)}\sum_{s=0}^{\infty}\frac{(-1)^s\,(M-1)\,(M+2s-1)\,\lambda^{2s}}{[(M-1)\,(M+1)\cdots(M+2s-1)]^2} \tag{23}$$

for $M = 0, 2, M \geq 4$,

$$c_3 = \frac{\lambda}{3} - \frac{4}{3}\,\frac{\lambda^3}{3!}\sum_{s=0}^{\infty}\frac{(-1)^s\,2(2s+2)\,\lambda^{2s}}{[2\cdot 4\cdot 6\cdots(2s+2)]^2}, \tag{24}$$

$$c_1 = -\frac{\lambda^3}{3!}\sum_{s=0}^{\infty}\frac{(-1)^s\,2\,(2s+2)\,\lambda^{2s}}{[2\cdot 4\cdot 6\cdots(2s+2)]^2}, \tag{25}$$

$$c_{-1} = -c_1;$$

and $g_1(\lambda)$ is a constant.

It can be determined by comparing the value of

$$\int_{-\infty}^{0} e^{\lambda y}\frac{dy}{(y^2+1)^{\frac{3}{2}}}$$

obtained from (22) with that obtained from the formula

$$\int_{-\infty}^{0} e^{\alpha y} (y^2 + c^2)^{-\frac{3}{2}} \, dy = \frac{\alpha}{c} \left[K_1 (\alpha c) - \frac{\pi}{2} Z_o' (\alpha c) \right] . \tag{26}$$

Where

$$Z_o' (x) = \frac{2}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{[1 \cdot 3 \cdot 5 \cdots (2s-1)]^2 (2s+1)} \tag{27}$$

$$K_1 (x) = -\frac{\pi}{2} N_1 (x) ,$$

And $N_1 (x)$ is Neumann's cylindrical harmonic of the first order.[1]

The formula for $g_1 (\lambda)$ is

$$g_1 (\lambda) = \frac{\lambda^2}{3} \left[K_1 (\lambda) - \frac{\pi}{2} Z_o' (\lambda) \right] - \frac{1}{3} - c_o . \tag{28}$$

The third method is best adapted to intermediate values of λ in the neighborhood of $\lambda = 1$.

By expansion into series we obtain

$$\int_{-\infty}^{u} e^{\lambda y} \frac{dy}{(y^2+1)^{\frac{1}{2}}} = \sum_{M=0}^{\infty} d_M u^M (u^2+1)^{-\frac{1}{2}} + d_{-1} \log \left(\sqrt{u^2+1} + u \right) + g_2 (\lambda) . \tag{29}$$

Where

$$d_M = \frac{\lambda^{M+1}}{(M+1)!} \cdot \frac{1}{(M-1)} \sum_{s=0}^{\infty} \frac{(-1)^s (M+1)(M+2s+1) \lambda^{2s}}{[(M+1)(M+3) \cdots (M+2s+1)]^2} \tag{30}$$

for $M = 0$, $M \geq 2$,

$$d_{-1} = \frac{\lambda^2}{2} \sum_{s=0}^{\infty} \frac{(-1)^s (2s+2) \lambda^{2s}}{[2 \cdot 4 \cdot 6 \cdots (2s+2)]^2} , \tag{31}$$

$$d_1 = 1 - d_{-1} ; \tag{32}$$

also

$$d_{M+2} = \frac{\lambda^{M+1}}{(M+2)!} - \frac{(M-1)}{(M+2)} d_M \tag{33}$$

for $M \geq 2$.

The constant of $g_2 (\lambda)$ can be determined in the same manner as $g_1 (\lambda)$ and is

$$g_2 (\lambda) = \lambda \left[K_1 (\lambda) - \frac{\pi}{2} Z_o' (\lambda) \right] - d_o . \tag{34}$$

[1] The notation here is that of the Jahnke-Emde "Funktionentafeln," p. 93, where $K_1(x)$ and $N_1(x)$ are defined and tabulated. Formula (26) is easily derived by differentiation from equation (2) in Nielsen's "Handbuch der Theorie der Cylinderfunktionen," p. 224, for the case $\upsilon = 0$.

The Fourth Method: This is convenient for small values of λ. By a combination of partial integration and expansion into series

$$e^{-\lambda u}\int_{-\infty}^{u} e^{\lambda y}\frac{dy}{(y^2+1)^{\frac{1}{2}}} = \frac{u}{\sqrt{u^2+1}}$$

$$-\lambda e^{-\lambda u}\left\{\sum_{M=0}^{\infty} f_M u^M\sqrt{u^2+1} + f_{-1}\log\left(\sqrt{u^2+1}+u\right) - g_3(\lambda)\right\}, \quad (35)$$

where

$$f_M = \frac{\lambda^M}{M!}\sum_{s=0}^{\infty}\frac{(-1)^s(M+2s+1)\lambda^{2s}}{[(M+1)(M+3)\cdots(M+2s+1)]^2} \quad (36)$$

for $M \geq 0$.

$$f_{-1} = -f_1;$$

Also

$$f_{M+2} = \frac{1}{M+2}\left\{\frac{\lambda^M}{M!} - (M+1)f_M\right\}. \quad (37)$$

Equation (35) can be put into the form

$$e^{-\lambda u}\int_{-\infty}^{u} e^{\lambda y}\frac{dy}{(y^2+1)^{\frac{1}{2}}} = \frac{u}{\sqrt{u^2+1}} - (1-e^{-\lambda u})\frac{\sqrt{u^2+1}}{u}$$

$$-\lambda e^{-\lambda u}\left\{\sum_{M=0}^{\infty} k_M u^M\sqrt{u^2+1} + f_{-1}\log\left(\sqrt{u^2+1}+u\right) - g_3(\lambda)\right\}, \quad (38)$$

where

$$k_M = \frac{\lambda^M}{M!}\sum_{s=1}^{\infty}\frac{(-1)^s(M+2s+1)\lambda^{2s}}{[(M+1)(M+3)\cdots(M+2s+1)]^2} \quad (39)$$

and

$$g_3(\lambda) = \left[K_1(\lambda) - \frac{\pi}{2}Z_o'(\lambda)\right] + f_o$$

All the series in the last three methods coverge for all values of λ and u.

The Fifth Method: For values of $\lambda > 1$ and too small to use the first method, the next three methods are not very satisfactory for the larger negative values of u. The following formula is better in this case.

Expanding the binomial factor,

$$e^{-\lambda u}\int_{-\infty}^{u} e^{\lambda y}\frac{dy}{(y^2+1)^{\frac{1}{2}}} = \sum_{M=0}^{\infty} a_M u^{-M} + a_{-1}e^{-\lambda u}Ei(\lambda u) \quad (40)$$

for $u < -1$,

Where [2]

$$a_M = -\sum_{s=1}^{\infty} \frac{(M-1)!\, l_{M+s} \lambda^{s-1}}{(M+s-1)!} \tag{41}$$

for $M \geq 1$,

$$a_0 = 0$$

$$a_{-1} = -\lambda a_1$$

And

$$l_M = 0$$

for M even;

$$l_M = \frac{(-1)^{\frac{M-1}{2}} (M-2)!}{2^{M-3}\left[\left(\frac{M-3}{2}\right)!\right]^2}$$

for M odd.

APPROXIMATIONS AND INTERPRETATION OF RESULTS.

(a) It can easily be shown that in the case when α and β are infinitesimal, (15) and (16) reduce to

$$i_1 = k_1\left\{\frac{u-c}{U_1} - \frac{u+c}{U_2}\right\}, \tag{42}$$

$$i_2 = -k_2\left\{\frac{u-c}{U_1} - \frac{u+c}{U_2}\right\}. \tag{43}$$

The shape of these curves is shown in Fig. 2 (a), which was plotted for $c = 1.174$.

(b) A second approximation.

Everything of importance happens in an interval $|u| < 10$ unless $c > 2$ or 3.

Therefore, if $\alpha, \beta < .01$,

$$e^{\alpha u} = 1 + \alpha u$$

$$e^{\beta u} = 1 + \beta u$$

for $|u| < 10$ to within less than 1 per cent. It can be shown that under these conditions equations (15) and (16) reduce to

$$i_1 = k_1\left\{\frac{u-c}{U_1} - \frac{u+c}{U_2} - (\alpha+\beta-\alpha_2)(U_1 - U_2 - 2c)\right\}, \tag{44}$$

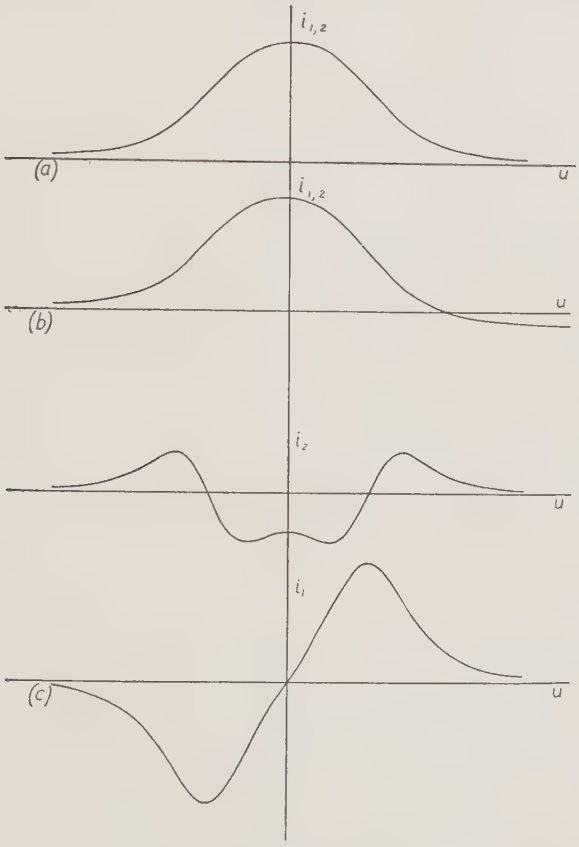
$$i_2 = -k_2\left\{\frac{u-c}{U_1} - \frac{u+c}{U_2} - (\alpha+\beta)(U_1 - U_2 - 2c)\right\}. \tag{45}$$

[2] $$Ei(x) = \int_{\infty}^{-x} \frac{e^{-x}dx}{x} = \int_{-\infty}^{x} \frac{e^{x}dx}{x}.$$

It is defined and tabulated in the Jahnke-Emde "Funktionentafeln," pp. 19–23.

Since both $(\alpha + \beta)$ and $(\alpha + \beta - \alpha_2)$ are positive, the i_1, u curve and the i_2, u curve have the same general shape, which is illustrated in Fig. 2 (b), which was plotted for $c = 1.174$.

Fig. 2.



(c) An approximation for large values of α and β.

By partial integration

$$e^{-\lambda u}\int_{-\infty}^{u} e^{\lambda y}\frac{dy}{(y^2+1)^{\frac{3}{2}}} = \frac{1}{\lambda}\frac{1}{(u^2+1)^{\frac{3}{2}}} + \frac{1}{\lambda^2}\cdot\frac{3u}{(u^2+1)^{\frac{5}{2}}} \qquad (46)$$

$$+\frac{3}{\lambda^2}\, e^{-\lambda u}\int_{-\infty}^{u} e^{\lambda y}\frac{(4y^2-1)}{(y^2+1)^{\frac{7}{2}}}\, dy.$$

The ratio, R_1, of the integrand of the last integral to that of the original integral is $R_1 = \frac{3}{\lambda^2}\frac{4y^2-1}{(y^2+1)^2}$

The maximum numerical value of R_1 is $R_1 = \frac{3}{\lambda^2}$. Hence if R_1 is to be less than 1 per cent., $\lambda > 18$.

Consequently for $\alpha, \beta > 18$

$$i_1 = \frac{k_1}{\alpha\beta}\left\{\alpha_2\left(\frac{1}{U_1^3} - \frac{1}{U_2^3}\right)\right. \tag{47}$$

$$\left. + \frac{3[\alpha_2(\alpha+\beta) - \alpha\beta]}{\alpha\beta}\left[\frac{u-c}{U_1^5} - \frac{u+c}{U_2^5}\right]\right\},$$

$$i_2 = \frac{3k_2}{\alpha\beta}\left\{\frac{u-c}{U_1^5} - \frac{u+c}{U_2^5}\right\}. \tag{48}$$

These equations show that in this case, neglecting a numerical factor, the i_1, u curve is only slightly different from the e.m.f. curve, and the i_2, u curve is the derivative of the e.m.f. curve. The general shape of these curves is shown in Fig. 2 (c), which was computed for α and β infinite and $c = 1.174$. The slight maximum in the i_2, u curve at $u = 0$ is due to the large value of c. As c decreases this becomes less and less prominent, and finally vanishes and becomes a minimum for $c < \frac{1}{2}\sqrt{3} = .866$.

Comparison of Figs. 2 (a), 2 (b), and 2 (c) shows the general trend of the phenomena as α and β increase from small values to large values. When $\alpha = \beta = 0$, the i, u curves for both primary and secondary are similar to Fig. 2 (a) and do not cross the u-axis. When α and β are small the i, u curves for both primary and secondary, shown in Fig. 2 (b), are similar in their general characteristics to the e.m.f. curve in that each has one maximum and one minimum, cross the u-axis and are zero for $u = \pm\infty$. The second numerical maximum is small, however, in comparison with the first. As α and β increase, the second numerical maximum increases with respect to the first, and it would seem as though there must be cases where both i, u curves are very similar indeed to the e.m.f. curve, though displaced along the u-axis. If α and β are increased still further, the i_1, u curve approaches coincidence, neglecting a numerical factor, with the e.m.f. curve, while the i_2, u curve begins crossing the u-axis a second time and approaches coincidence, again neglecting a numerical factor, with the time derivative of the e.m.f. curve. These conclusions are based on a consideration of the approximation cases (a), (b), and (c), and must not be interpreted too rigorously.

(*d*) The case when $\beta = 0$.

If either $\alpha_1 = 0$ or $\alpha_2 = 0$, then from (14) $\beta = 0$, and from (16)

$$i_2 = -k_2 [I(\alpha, u - c) - I(\alpha, u + c)]. \tag{49}$$

If α is small the i_2, u curve has in this case the shape shown in Fig. 2 (*b*). If α is very large, however, it can be shown just as in case (*c*) that

$$i_2 = -\frac{k_2}{\alpha}\left\{\frac{1}{U_1^3} - \frac{1}{U_2^3}\right\} = -\frac{k_2}{\alpha}\varphi(u), \tag{50}$$

which shows that disregarding the numerical factor, the i_2, u curve coincides with the e.m.f. curve.

It follows that when $\beta = 0$, the i_2, u curve cannot cross the axis more than once.

TIME LAGS.

If t_{o1} and t_{o2} are values of t such that

$$i_1 = 0 \text{ when } t = t_{o1},$$
$$i_2 = 0 \text{ when } t = t_{o2}$$

we may call t_{o1} and t_{o2} the time lags of primary and secondary, respectively. Let u_{o1} and u_{o2} be the corresponding values of u. From (3)

$$t = \frac{b}{V}u. \tag{51}$$

From (15) and (16) we saw that the shape of the i, u curves was a function of α_1, α_2, k, and c. Hence u_{o1} and u_{o2} are functions only of α_1, α_2, k, and c. It follows then from (13) and (51) that varying the factors in α_1 and α_2 while keeping α_1, α_2, k, and c constant, we can vary the time lags. From (51), if we take $\frac{b}{V}$ small and $\frac{R_1}{L_1}$ and $\frac{R_2}{L_2}$ large, the time lags will be relatively small; while if we take $\frac{b}{V}$ large and $\frac{R_1}{L_1}$ and $\frac{R_2}{L_2}$ small, the time lags will be relatively large. This is what one would expect independently of the mathematics.

PART II.

THE EFFECT OF OBLIQUITY AND ECCENTRICITY OF THE TRAJECTORY ON THE E.M.F. CURVE.

Equations (1) and (4) were derived for the case when the magnet, moving in the direction of its axis, passes perpendicularly through the centre of the coil. It can be shown that if the

magnet moving in the direction of its axis passes perpendicularly through the coil at a distance d from the centre of the coil, the e.m.f. is given to the third order of approximation by the expression,

$$E = \frac{p}{b^3}\left\{\frac{1}{U_1^3}\left[1+\left(\frac{15}{4}\frac{1}{U_1^2}-3\right)\frac{g^2}{U_1^2}\right]\right. \tag{52}$$
$$\left.-\frac{1}{U_2^3}\left[1+\left(\frac{15}{4}\frac{1}{U_2^2}-3\right)\frac{g^2}{U_2^2}\right]\right\},$$

where $g=\frac{d}{b}$. This reduces to (4) for $g=0$ and is only slightly different from (4) if $g<.1$.

It can also be shown that if the magnet moving in the direction of its axis passes through the centre of the coil along a line making an angle θ with the axis of the coil, the e.m.f. to the third order of approximation is

$$E = \frac{p}{b^3}\left\{\frac{1}{U_1^3}\left[1-\frac{1}{4}\sin^2\theta+\frac{15}{4}\frac{(u-c)^2\sin^2\theta}{U_1^4}\right]\right.$$
$$\left.-\frac{1}{U_2^3}\left[1-\frac{1}{4}\sin^2\theta+\frac{15}{4}\frac{(u+c)^2\sin^2\theta}{U_1^4}\right]\right\}. \tag{53}$$

The expressions in brackets differ from unity by less than 1 per cent. if $\sin\theta<.05$.

Finally, it can be shown that if the trajectory of the centre of the magnet passes perpendicularly through the centre of the coil and the magnet is inclined to its trajectory by an angle θ', the e.m.f. to the second order of approximation is

$$E = p\left\{\frac{1}{T_1^3}\left[1+\left(\frac{15}{4}\frac{b^2}{T_1^2}-3\right)\frac{a^2\sin^2\theta'}{T_1^2}\right]\right.$$
$$\left.-\frac{1}{T_2^3}\left[1+\left(\frac{15}{4}\frac{b^2}{T_2^2}-3\right)\frac{a^2\sin^2\theta'}{T_2^2}\right]\right\} \tag{54}$$

Where

$$T_1=\sqrt{(Vt-a\cos\theta')^2+b^2}$$
$$T_2=\sqrt{(Vt+a\cos\theta')^2+b^2}.$$

This shows that the effect of the tilt is small for $\sin\theta'<.1$, if $\frac{a}{b}=c<$ about 1.

The more general case, the one most likely to occur in practice, where the magnet making an angle with its trajectory passes through the coil eccentrically along a line making an angle with the axis of the coil, has been investigated by the author. The

resulting formula for the e.m.f. is of the second order in the deviating quantities, indicating that the errors due to small deviations are negligible.

Experimental tests were carried out and the theory of Part I verified to a satisfactory degree of approximation.

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